Spectral Variability in Fixed Windows using Fractional Fourier Transform: Application in Power Spectral Density Estimation

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Abstract: In statistical signal processing, power spectral density estimation is a frequency domain analysis in which power contents of a signal are measured with respect to frequency components of that signal. The power estimation of a signal can be carried out more precisely by using a window with a narrower 3-dB bandwidth and higher side-lobe attenuation. Theoretically, these two spectral parameters show trade-off in variable windows and remain constant in fixed windows. In this work, spectral behavior of fixed windows has been elaborated using Fractional Fourier Transform (FRFT) keeping their inherent time domain behavior intact. The FRFT is an extension of conventional Fourier transform with an additional variable parameter, known as rotation angle, which makes it more flexible and useful in various signal processing applications viz. power estimation and designing of tunable transition band FIR filters. In this article, variability in 3-dB bandwidth and sidelobe attenuation of fixed windows has been achieved by exploiting the available flexibility in FRFT and obtained variability has been applied in the estimation of signal power. Simulation results demonstrate that both of these two spectral parameters are improved and hence, trade-off problem between resolution and spectral leakage in the power spectral density estimation is overcome upto an extent.

Keywords: Spectral Variability, Fractional Fourier Transform, Power Spectral Density Estimation, Fixed Windows.

I. INTRODUCTION

The spectral estimation is the most important characteristic to calculate Power Spectral Density (PSD) for analysis and processing of random signals and power signals [1]. The PSD is theoretically computed by using the signal observed in the infinite-time duration. However, practically it is impossible to use such an infinite-time signal; rather only a finite-time signal is taken into the consideration. This is equivalent to a truncation of infinite-time signal to finite length by applying a rectangular window. Therefore, the estimated PSD can be considered convolution between the theoretical PSD and the spectrum of a rectangular window in frequency domain. In other words, it can be said that the accuracy of the estimated PSD is directly correlated by the window that is used for PSD estimation.

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In the conventional Fourier transform (FT) based spectrum of window functions, the side-lobe attenuation (SLA), 3-dB bandwidth, and side-lobe fall-off rate (SLFOR) are three performance measuring parameters for PSD estimation [2,3,4]. The SLA is the difference between the magnitude of the main-lobe and the maximum side-lobe level (MSLL). The SLFOR is the asymptotic decay rate of side-lobe peaks. Better resolution of the estimated PSD can be obtained if 3-dB bandwidth is reduced. Spectral leakage [5,6,7] can be reduced by increasing the SLA and SLFOR. Therefore, an ideal window for PSD estimation has zero bandwidth and infinite SLA like impulse function in the frequency domain. Conventional windows, such as Kaiser, Gaussian, Dolph-Chebyshev, and so on [2, 8,9,10,11,12,13], are generally able to control the 3-dB bandwidth or SLA by one variable parameter; while fixed windows, e. g. rectangular (RW), generalized Hamming (GHW; Hanning and Hamming), Blackman (BW), and Triangular (TW), have all these three parameters constant. This causes a fixed resolution and spectral leakage in the estimated PSD of a signal. Therefore, it is desirable to have some control over 3-dB bandwidth and SLA of fixed windows spectrum. The fixed windows with fractional Fourier transform (FRFT), which has got several recent applications in signal processing [14,15,16,17,18,19,20,21], are expected to overcome this problem. FRFT which is a generalization of the FT with an additional degree of freedom, better known as FRFT order 'a', transforms a time domain signal into variable fractional domain in place of fixed frequency domain. In this work, discrete version of FRFT [22] has been used for the estimation of PSD of a discrete signal using discrete fixed windows.

The continuous-time FRFT of a signal $x(t)$ is given as [16]-

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\, dt \tag{1}$$

where, $\alpha = a\pi / 2$ is the rotation angle of the transformed signal and $K_\alpha(t, u)$ is the transformation kernel of the FRFT is defined as-

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1 - i\cot(\alpha)}{2\pi}}\, e^{i\left(\frac{u^2+t^2}{2}\right)\cot(\alpha) - iut\,\mathrm{cosec}(\alpha)} & \text{if } \alpha \neq n\pi \\ \delta(t-u) & \text{if } \alpha = 2n\pi \\ \delta(t+u) & \text{if } (\alpha+\pi) = 2n\pi \end{cases} \tag{2}$$

In this paper, FRFT based analysis of all four fixed windows has been done and it has been observed that their spectral parameters can be made variable upto some extent.

It has also been observed that spectral characteristics of these windows get reversed after certain window length (has been termed as 'limiting length') while varying FRFT order 'a' from 1 to 0. A generalized relationship for the limiting length of fixed windows in terms of window length and padded zeros has been established. This relationship has been utilized to introduce spectral variability in fixed windows which is applied PSD estimation.

The rest of the paper is organized as follows. In Section II, brief introduction about PSD estimation is presented. Section III presents the establishment of relationship for limiting length. Results and performance analysis has been included in Section IV. Finally, Section V concludes this paper.

II. POWER SPECTRAL DENSITY ESTIMATION

The PSD of a signal $x(t)$ is the FT of its autocorrelation. Therefore, it can be assumed that estimating the PSD is equivalent to estimating the autocorrelation. Mathematically, PSD $P_x(f)$ is obtained by-

$$P_x(f) = \int_{-\infty}^{\infty} r_x(\tau)e^{-2\pi f\tau}\, d\tau \tag{3}$$

where $r_x(\tau)$ is the autocorrelation function of signal $x(t)$. In practice, it is not possible to use infinite duration signal. Therefore, PSD of a finite-time signal is obtained by the truncation of infinite duration time domain signal using a suitable window function, which is denoted in [23, 24]

as- $$\bar{P}_x(f) = \int_{-\infty}^{\infty} w(\tau)\, r_x(\tau)e^{-2\pi f\tau}\, d\tau \tag{4}$$

where, $w(\tau)$ is window function of duration $\left\{-\frac{T}{2} to\ \frac{T}{2}\right\}$.

In FT property, multiplication of two functions in time domain corresponds to convolution in frequency domain. Thus, the PSD can also be defined as-

$$\bar{P}_x(f) = W(f) \circledast R_x(f) \tag{5}$$

where, '$\circledast$' denotes convolution operation while $W(f)$ and $R_x(f)$ are the spectrums of window and autocorrelation function respectively. Therefore, it can be stated that the frequency resolution and spectral leakage in PSD estimation solely depend on the spectral properties of used window function.

In this work, estimation of PSD has been carried out using FRFT to exploit its additional features over FT. Therefore, relationship (5) has been modified as-

$$\bar{P}_x(f) = W_\alpha(f) \circledast R_x(f) \tag{6}$$

where, $W_\alpha(f)$ is the fractional domain spectrum of a window used in PSD estimation.

III. CALCULATION OF LIMITING LENGTH FOR FIXED WINDOWS

The FRFT is an identity operator when $a = 0$, whereas, it maps the signal into frequency domain for $a = 1$. Thus, if the FRFT order a lies between 0 and 1 signal will be composed of frequency and time components both. In FT domain, the null bandwidth (NBW; position of first null in the spectrum of a window function) is having inverse relationship to the window length and the product of these two parameters remains constant for every window function. This property of window functions can be described by the uncertainty principle [25], according to which, a function cannot be highly concentrated in both time and frequency domains. If the window length is small the NBW is large and at a particular window length these two parameters become equal and for further increments in the window length NBW becomes smaller than the window length. This property of window functions has been exploited in this work to analyze the behavior of windows about this particular window length in fractional Fourier domain. It has been observed in fractional domain simulation studies that the NBW shrinks for small window length and expands for large window length when the FRFT order is varied from 1 to 0. This behavior is justified by the uncertainty principle because the FRFT plane moves towards time axis when 'a' is varied from 1 to 0. Thus, the FRFT provides another parameter to vary the NBW without varying the window length.

Simulation studies establish that the time spread of discrete windows padded with M zeros equals the main-lobe width of window function at a 'limiting length' (N_L) for $a = 1$. To find this limiting length N_L, time and frequency domain spreads of four fixed windows have been shown in Figures 1-4. The x-axis of both the domains of N sampled and M zero padded windows has been normalized in the same range i.e. -0.5 to 0.5, however, they have been shown only in the range of -0.1 to 0.1 for the sake of clarity. It can be seen from Figure 4.7 that the non-zero time spread of half of the RW is given by (N+1)/2 and $N/2$ for N odd and even respectively. The corresponding NBW of RW is given by (M+N)/N. This equality for a RW results from the following facts-

(i) Half width of window in time domain, in terms of number of samples is (N+1)/2 (for N odd) or $N/2$ ((for N even).
(ii) In FT based spectrum, NBW in continuous domain is fixed i.e., $2\pi/\tau$ [26].
(iii) In FRFT based spectrum, NBW in continuous domain varies with 'a' as $\frac{2\pi}{\tau} Sin\left(\frac{a\pi}{2}\right)$ [27].
(iv) In discrete domain τ equals N, so the NBW in discrete domain equals $2\pi/N$.
(v) In frequency domain 2π corresponds to the total number of samples in a function. Hence, for a window width of N samples padded with M zeroes, 2π in frequency domain equals N+M.
(vi) So, the NBW width of a RW of length N padded with M zeroes in discrete domain is given by (M+N)/N.

Thus, the generalized expression for the limiting length, N_L, of fixed windows at which half width of the window function in time domain equals the NBW in frequency domain can be obtained using above mentioned facts for $a = 1$, as-

$$\frac{N+1}{2} = \frac{(M+N)p}{N} \qquad if\ N\ is\ odd$$
$$\frac{N}{2} = \frac{(M+N)p}{N} \qquad if\ N\ is\ even \tag{7}$$

where, M equals number of padded zeroes on either sides of the window function to increase the resolution. Since NBW for GHW & TW is twice that of NBW of RW and thrice for BW [5]. Therefore, p = 1 for RW, 2 for GHW & TW and 3 for BW.

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The mathematical relation given in (7) is a quadratic equation and its positive root, defines the 'limiting length' of window functions, which is given as-

$$N_L = p + p\sqrt{1 + \frac{2M}{p}} \quad (8)$$

Table-1 includes the calculated values of limiting lengths, N_L, using (8) for four fixed window functions. The time and frequency spread of these windows about the calculated limiting lengths are shown in Figures 1, 2, 3 and 4. These simulation results also support the analytical relationship for N_L given by (8).

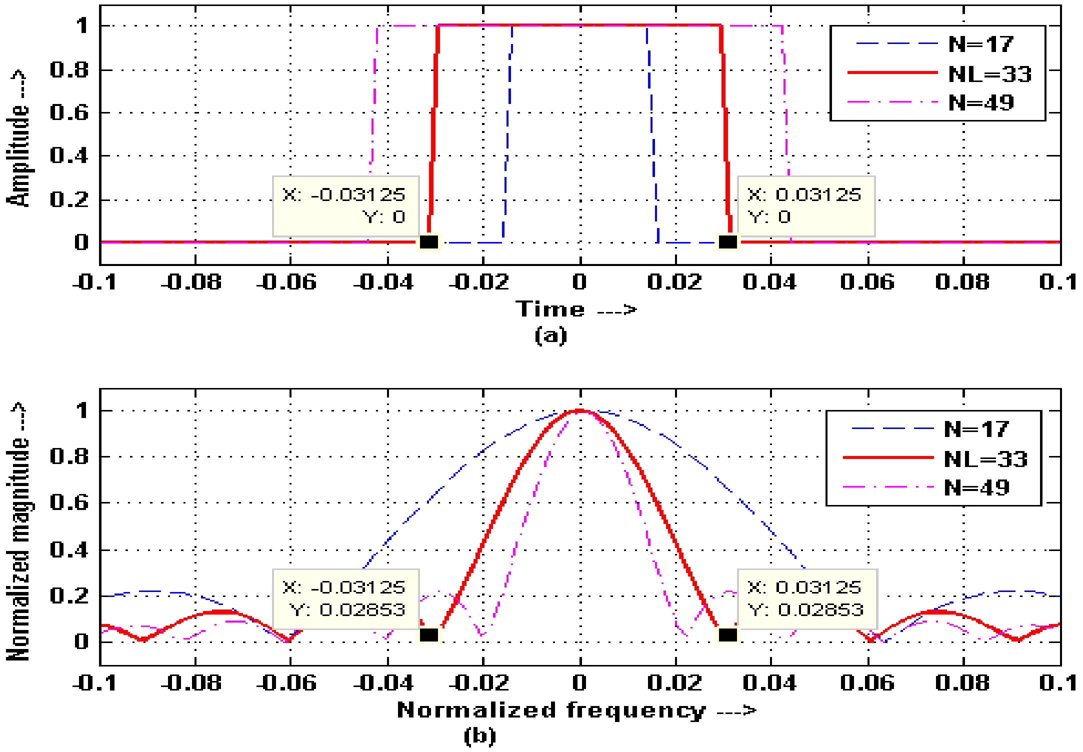

Figure 1: Expanded view of RW in (a) time domain (b) frequency domain for different lengths *N* and *M* = 512 with *a* = 1.

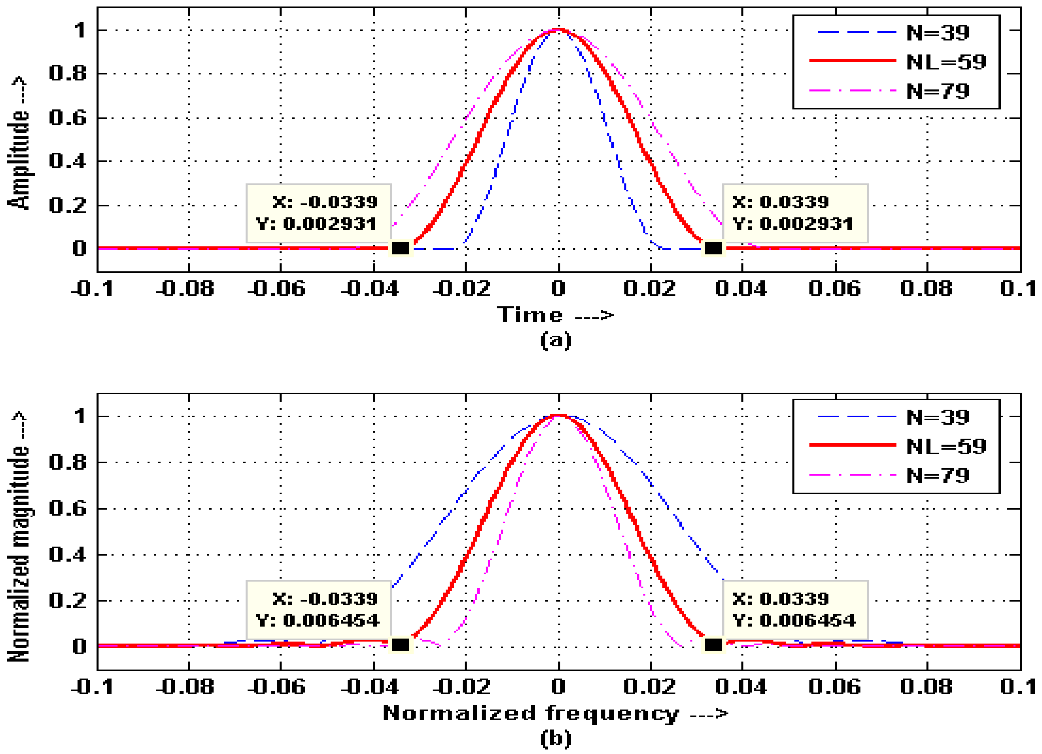

Figure 2: Expanded view of Hanning window in (a) time domain (b) frequency domain for different lengths *N* and *M* = 768 with *a* = 1.

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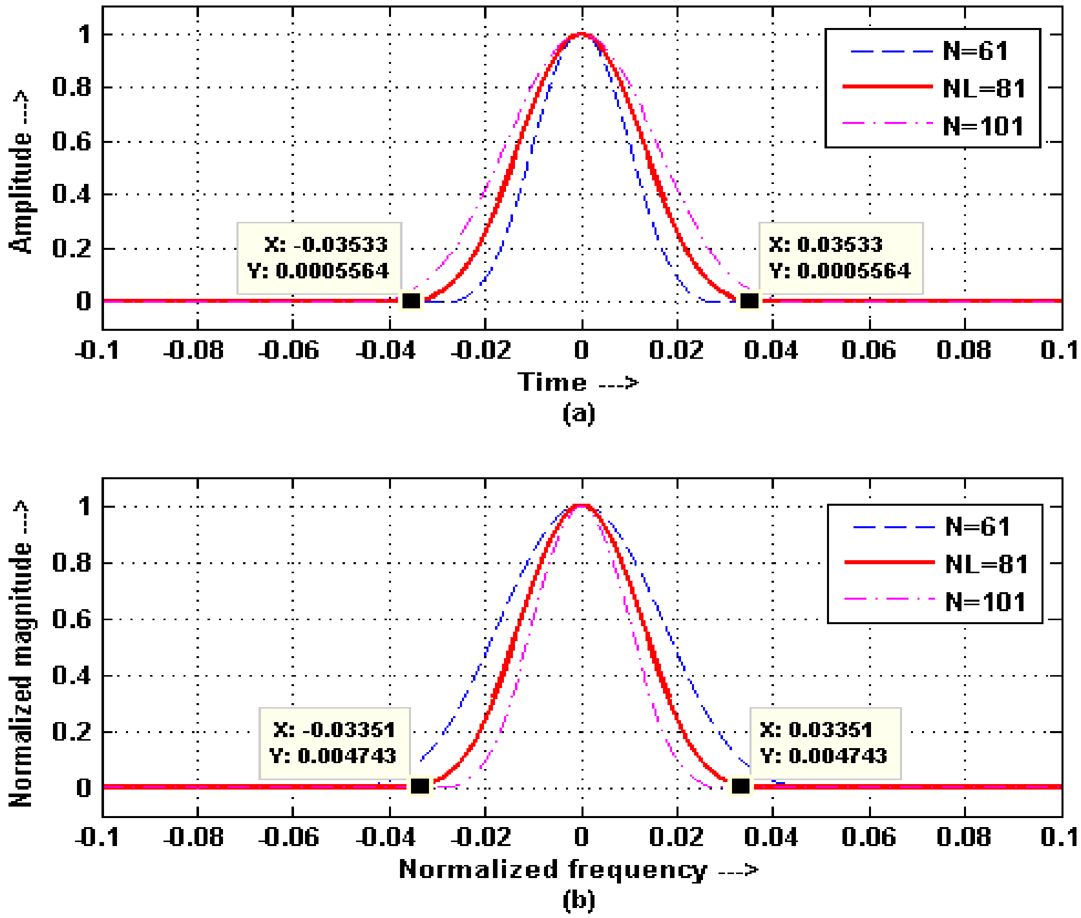

Figure 3: Expanded view of BW in (a) time domain (b) frequency domain for different lengths *N* and *M* = 1024 with *a* = 1.

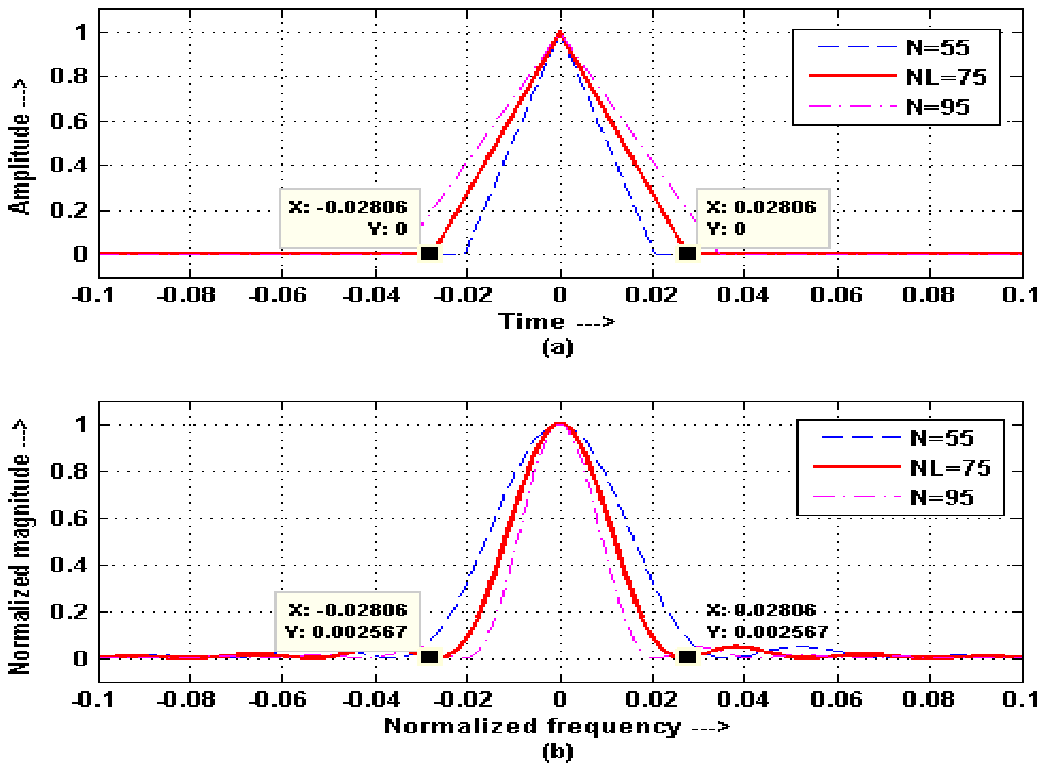

Figure 4: Expanded view of TW in (a) time domain (b) frequency domain for different lengths *N* and *M* = 1280 with *a* = 1.

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Spectral variability of these four fixed windows about their limiting lengths is shown in Figures 5, 6, 7 and 8. It is observed that when window length N is less than N_L, the main-lobe lobe width of the windows shrinks as FRFT order is reduced from 1 to 0, while for $N > N_L$ the main-lobe exhibits a reverse behavior for the same variation in FRFT order.

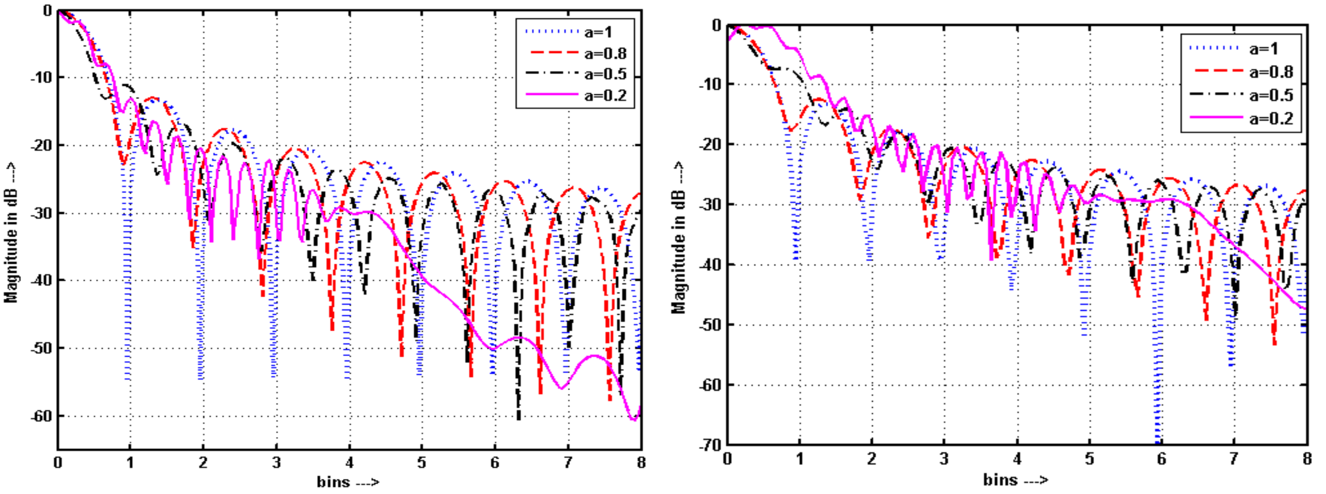

Figure 5: Spectral variability in RW with different 'a' for (i) N = 27 (< N_L) and M = 512 (ii) N = 37 (> N_L) and M = 512.

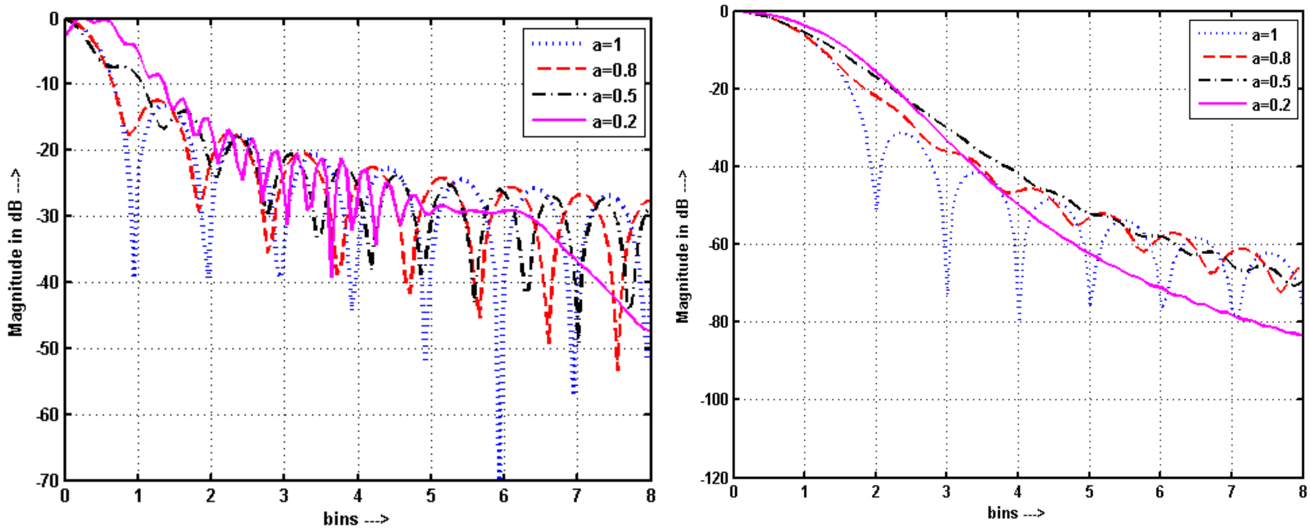

Figure 6: Spectral variability in HW with different 'a' for (i) N = 50 (< N_L) and M = 768 (ii) N = 70 (> N_L) and M = 768.

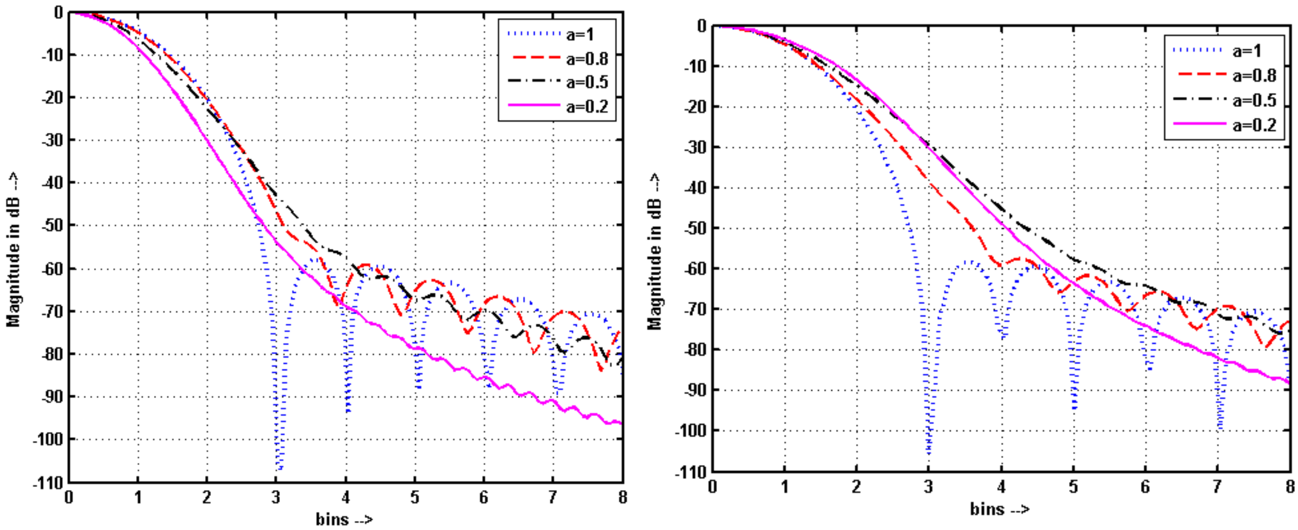

Figure 7: Spectral variability in BW with different 'a' for (i) N = 70 (< N_L) and M = 1024 (ii) N = 90 (> N_L) and M = 1024.

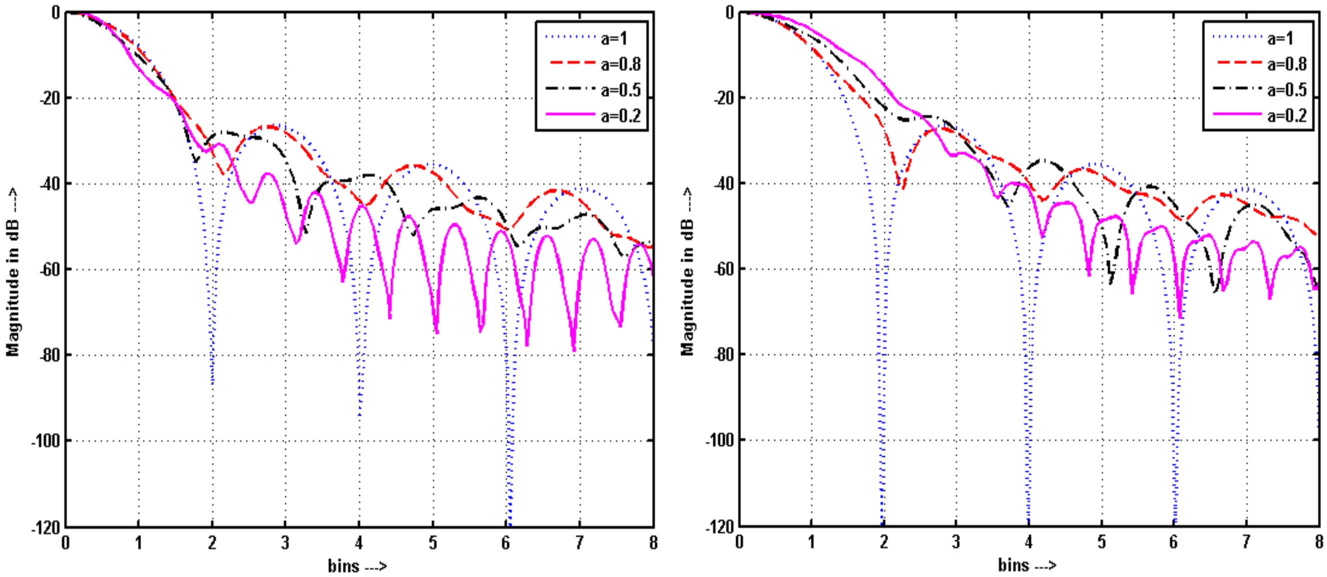

Figure 8: Spectral variability in TW with different 'a' for (i) N = 61 (< N_L) and M = 1280 (ii) N = 81 (> N_L) and M = 1280.

IV. SIMULATION AND PERFORMANCE ANALYSIS

In this paper, PSD of an example signal $x(t) = 5\cos(30\pi t) + 2.5\cos(50\pi t) + 1.25\cos(70\pi t)$ has been estimated by fixed windows using FRFT. The signal has been sampled with sampling frequency of 100 Hz. Therefore, the signal contains three normalized frequency components at 0.15, 0.25 and 0.35. The estimated PSD in fractional domain using RW, HW, BW, and TW are shown in Figures 9, 10, 11 and 12 respectively with window lengths less than their limiting lengths and different number of zeroes padded. It can be observed from Figure 9 that all three frequency components has negligible resolution in the estimated PSD using RW with FRFT order a = 1 because of wide 3-dB bandwidth and lower SLA. These two performance parameters get improved as FRFT order 'a' is reduced from 1 to 0.1 as shown in Figure 9. Similar behavior has also been observed for other three fixed windows as shown in Figures 10-12. The comparative performance of estimated PSD using fixed windows in factional domain is included in Table-2.

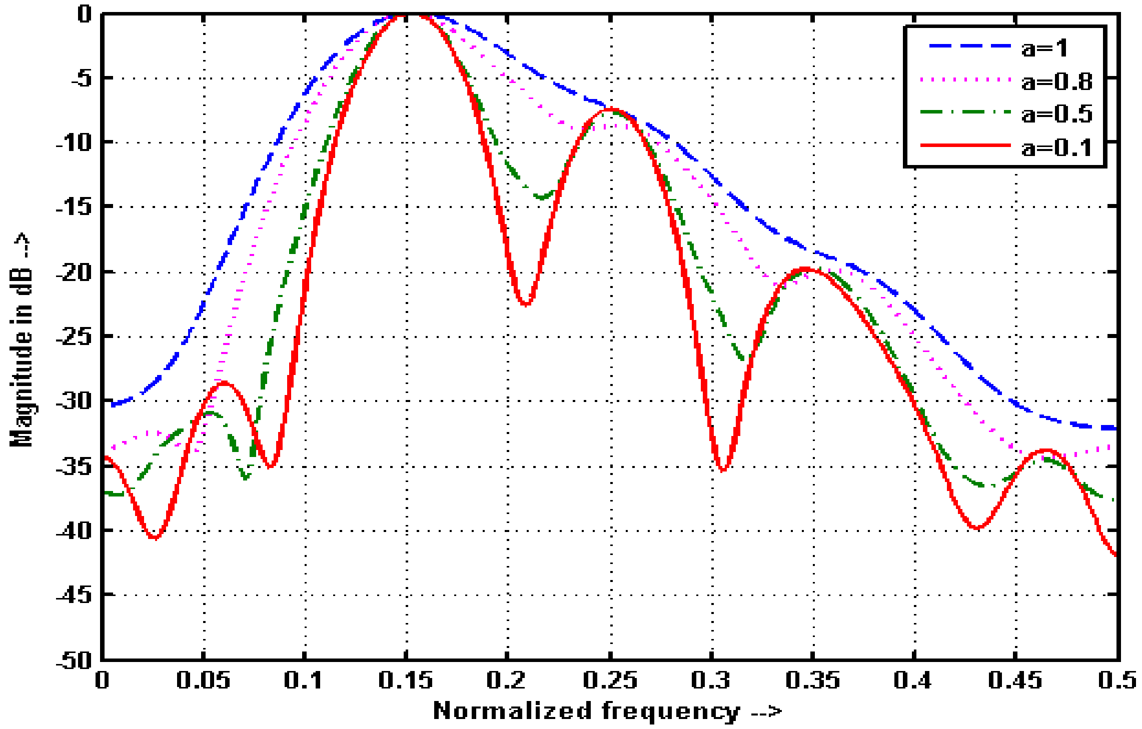

Figure 9: Estimated PSD using RW with different 'a' for N=27 (< N_L) and M = 512.

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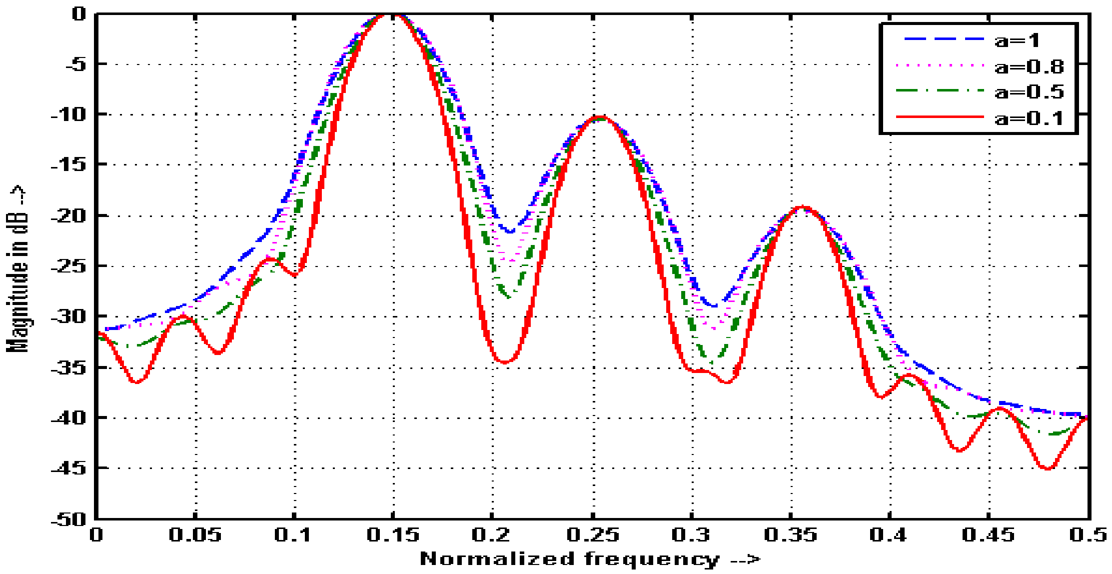

Figure 10: Estimated PSD using HW with different 'a' for N =43 (< N_L) and M = 768.

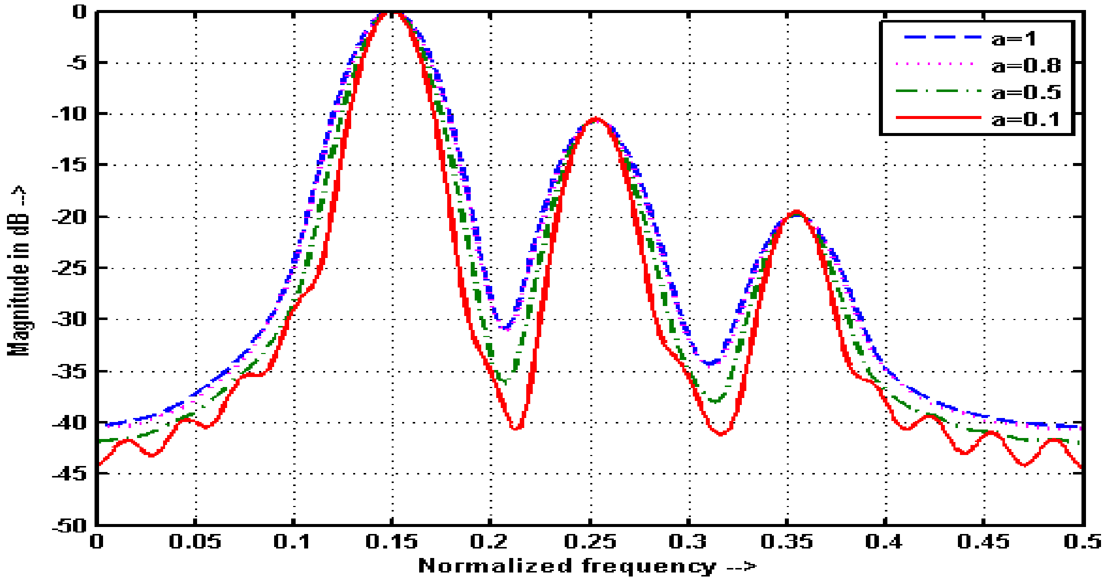

Figure 11: Estimated PSD using BW with different 'a' for N = 53 (< N_L) and M = 1024.

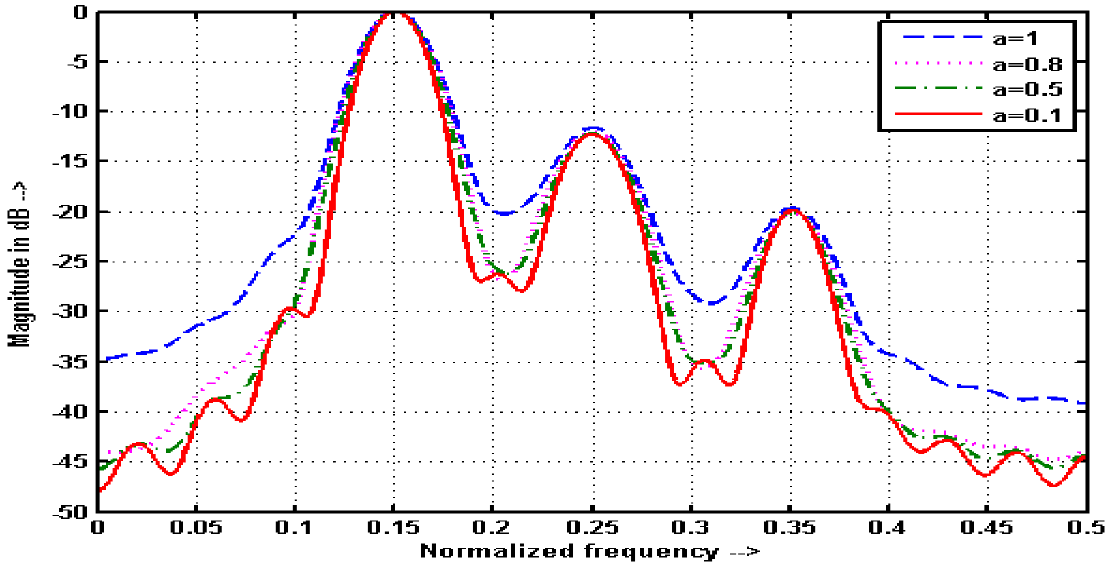

Figure 12: Estimated PSD using TW with different 'a' for N = 61 (< N_L) and M = 1280.

Table-1: Limiting Lengths for Fixed Window Functions

Window Type	Number of padded zeroes, *M*	Limiting length, N_L
RW	512	33
GHW	768	59
BW	1024	81
TW	1280	75

Table-2: Comparative Spectral Characteristics of Fixed Windows in Factional Domain

Window Type	3-dB bandwidth		MSLL (dB)	
	$a = 1$	$a = 0.1$	$a = 1$	$a = 0.1$
RW	0.068	0.038	-19.21	-28.42
HW	0.048	0.038	-21.10	-24.36
BW	0.033	0.024	-22.72	-27.71
TW	0.031	0.023	-22.79	-29.71

V. CONCLUSION

In this article, additional degree of freedom in FRFT has been exploited to achieve frequency domain behavioral variability in commonly used fixed windows which are considered invariable in both the domains. This work has got success to attain some extent of spectral variation and improvement in the performance parameters of fixed windows i.e. 3-dB bandwidth and side-lobe attenuation. Obtained spectral variability has been found useful to propose the FRFT based approach in PSD estimation which is supposed to solve trade-off problem between these two parameters. Simulation results clearly show that the proposed method offers enhanced resolution along with reduced spectral leakage simultaneously.

DECLARATION

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Ethical Approval and Consent to Participate	No, the article does not require ethical approval and consent to participate with evidence.
Availability of Data and Material/ Data Access Statement	Not relevant.
Authors Contributions	I am only the sole author of the article.

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